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On a Three-valued Logical Calculus and Its Application to the Analysis of the Paradoxes of the Classical Extended Functional Calculus

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Translator's Summary

A three-valued propositional logic is presented, within which the three values are read as 'true', 'false' and 'nonsense'. A three-valued extended functional calculus, unrestricted by the theory of types, is then developed. Within the latter system, Bochvar analyzes the Russell paradox and the Grelling-Weyl paradox, formally demonstrating the meaninglessness of both.

1. Translator's introduction

Dmitri Anotolevich Bochvar was born on 7 August 1903 in Moscow. He graduated from the Moscow Higher Technical School in 1924, received his doctorate in chemistry in 1950, and since 1953 has been working in mathematical logic and foundations of mathematics in the All-Union Institute of Scientific and Technical Information in Moscow. The article published in translation here first appeared in *Mathematicheskii sbornik*, 4 (46) (1937), 287–308.

Bochvar was one of the pioneers of many-valued logic; and in that field he is best known for two systems of three-values propositional connectives, the 'internal' and 'external' systems, which have their foundation in a distinction between two modes of statement assertion. The 'internal' system is functionally equivalent to Kleene's system of 'weak' connectives,[1] and the 'external' assertion connective is similar to Frege's 'horizontal'—in so far as both uniformly yield true or false assertions.[2] Bochvar develops the two systems as the basis for a three-valued extended functional calculus in which certain logical and semantic paradoxes may be resolved through a formal proof of their 'meaninglessness'.

Proof that Bochvar was successful in this project awaited articles appearing in

1 S.C. Kleene, *Introduction to metamathematics* (New York: D. van Nostrand, 1952).

2 See H.G. Herzberger, 'Truth and modality in semantically closed languages', in R.L. Martin (ed.) *The paradox of the liar* (New Haven: Yale, 1970), 25–46.

1944 and afterward, in which he worked on the problem of the consistency of this functional calculus and closely related systems.[3] He later investigated extensions of his functional calculi, thus addressing the question which Alonzo Church raised in review of the present article:

> . . . the suggested alternative to the theory of types is far from devoid of interest. The major question, it would seem, is . . . whether it is possible to obtain along these lines a system adequate to the purposes for which the extended functional calculus is usually employed, e.g., to the theory of finite cardinal numbers or to analysis—Bochvar does not address this point.[4]

Bochvar's recent published work has been in the areas of set theory and logic, including investigations of systems of many-valued logic.

Beyond application to paradoxes, many-valued logics have also been used to analyze other recalcitrant linguistic and logical phenomena. Bochvar's three-valued propositional systems have found their place in the literature on future contingent statements, on presuppositions, and on sortal incorrectness—as well as in the literature on the paradoxes.[5]

2. Notes on the translation and editing

In Bochvar's English résumé which follows his article, the Russian word *vyskazyvaniye* was rendered 'outsaying'. In conformity with one established usage in English-language logical writings, 'statement' is used consistently in place of *vyskazyvaniye*; and *predlozheniye* is translated as 'proposition'.

Bochvar used three distinct expressions to characterize nonsensical or meaningless statements: *bessmyslennoye vyskazyvaniye, bessoderzhatelnoye vyskazyvaniye*, and *vyskazyvaniye, ne imeyushcheye smysla*. Since he regarded the expressions as interchangeable, only the cognates of two expressions ('nonsense' and 'meaningless') are used in the translation, and choices between the two have been guided solely by considerations of English sentence construction.

Bochvar, like some of the authors whose works he cites, was not always careful to distinguish between talk of expressions and talk of their semantic interpretations. In several places, the distinction has been drawn in the translation where it was not made in the original.

For ease of reading, all the footnotes, both original and editorial, have been combined into one system. All editorial comments are enclosed within square

3 'K voprosu o neprotivorechivosti odnovo trexznachnovo ischislenia' ('On the consistency of a three-valued calculus'), *Matematicheskii sbornik*, 12 (54) (1943), 353–369; and 'K voprosu o parodoksa matematicheskoi logiki i teorii mnozhestva' ('On the paradoxes of mathematical logic and set theory'), *Matematicheskii sbornik*, 15 (57) (1944), 369–384.

4 *The journal of symbolic logic*, 5 (1940), 119.

5 See the surveys in S. Haack, *Deviant logic* (Cambridge: Cambridge University Press, 1974) and N. Rescher, *Many-valued logic* (New York: McGraw Hill, 1969). Rescher's book also contains a thorough discussion of the expressive powers of Bochvar's propositional systems, as well as comparisons with other many-valued systems.

brackets, except that the references have been silently converted to the house style of this publication (and also completed in those cases where Bochvar's references were only partial).

ON A THREE-VALUED LOGICAL CALCULUS AND ITS APPLICATION TO THE ANALYSIS OF THE PARADOXES OF THE CLASSICAL EXTENDED FUNCTIONAL CALCULUS

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The three-valued system which is the subject of this essay is of interest as a logical calculus for two reasons. First, it is developed through the formalization of a series of fundamental and self-evident relations between the statement predicates truth, falsehood, and nonsense, thus allowing a clear interpretation of an entirely logical character. Second, in this system the specifically logical problem of the analysis of the paradoxes of classical mathematical logic is solved by the method of formally proving the meaninglessness of certain statements.

The work has three parts. In the first, on the basis of certain semantic considerations, the elementary part of the system is developed—the statement calculus. In the second, a "restricted" functional calculus which corresponds to the statement calculus is briefly developed. Finally, in the third part, an analysis of the paradoxes of classical mathematical logic is given on the basis of a certain "extended" functional calculus.

I deeply acknowledge here my debt to Prof. V.E. Glivyenko for a number of valuable comments and suggestions. In particular, he suggested the most expedient form of the definition of $\bar{a}$ (I, §2, #1).

I
THE STATEMENT CALCULUS

§1. ELUCIDATION OF THE FUNDAMENTAL FEATURES OF THE STATEMENT CALCULUS ON THE BASIS OF CERTAIN SEMANTIC CONSIDERATIONS

In order to understand the fundamental features of the statement calculus, we submit the properties of fundamental types of statements to semantic analysis.

First, however, we define exactly the relation of the concepts "statement" [*vyskazyvaniye*] and "proposition" [*predlozheniye*]. We will say (in accordance with accepted usage) that a statement is meaningful if it is true or false. Further, we will call a statement a proposition if, and only if, it is meaningful. Statements which are not meaningful will be called meaningless or simply nonsensical. It is obvious that a proposition is a particular kind of statement. Every statement is either meaningless, true, or false. If a given statement A is meaningless, then the statements "A is false" and "A is true" are meaningful and false. The predicates truth, falsehood, and nonsense can be meaningfully applied to any statement.

Let A and B be arbitrary statements. Consider the following types of statements:

I	II
"A",	"A is true",
"not-A",	"A is false",
"A and B",	"A is true and B is true",

"*A* or *B*", "*A* is true or *B* is true",
"if *A*, then *B*", "if *A* is true, then *B* is true".

Let us agree to call type I and type II, respectively, internal and external forms of assertion,[6] denial, logical sum, logical product and implication. Obviously, the form "*A* is meaningless", which does not correspond to any internal form, is also associated with the group of external forms.

The semantic difference between any internal and corresponding external form is clear. It is easy to illustrate the principal difference between internal and external forms through the substitution of a meaningless statement for *A* (or *B*) in each. We turn first to the internal forms. It is obvious, that if *A* is a meaningless statement, then "not-*A*" is also meaningless; it is also clear that each combination of a meaningless statement *A* with any statement *B* by means of the operations "____ and ____", "____ or ____", and "if ____, then ____" will only result in a new meaningless statement.

Things stand quite differently with the external forms. Again let *A* be a meaningless statement. Then, obviously, its external assertion "*A* is true" is false and not meaningless. Similarly, an external denial—"*A* is false"—is false and not meaningless, if the statement *A* is meaningless. It is easy to see, that the remaining external forms likewise never result in meaningless statements when a meaningless statement is substituted for *A* in them.

Indeed, these external forms (logical sum, logical product, and implication) represent nothing other than the corresponding internal forms, in which *A* and *B* have been replaced by their external assertions. But as the external assertions never result in meaningless statements, the same must hold also of external logical sums, external logical products, and external implication.

It is obvious that for propositions the external forms are formally equivalent to the corresponding internal forms. This means that for propositions the corresponding internal and external forms are either simultaneously true or simultaneously false.

From this follows, in part, an explanation of the duality of the semantic interpretation of elementary functions of propositional variables in the classical calculus, which is common in the literature on mathematical logic.[7] Namely, together with the internal forms, external forms (for denial, logical sum, logical product, and implication) are also employed. (See, for example, *Principia mathematica*, Vol. 1, Part 1, Section A.) This duality of semantic interpretation, however, does not correspond to the actual nature of the formal classical propositional calculus. For the classical propositional calculus does not treat assertion as a function of propositional variables—in other words, it introduces only internal assertion—and thus its semantic interpretations are formulated on the basis of the system of internal forms.

6 An internal assertion is regarded as identical with the statement itself.

7 In this work 'classical propositional calculus' is understood specifically to mean a matrix calculus adequate to the propositional calculus of Hilbert and Ackermann (*Grundzüge der theoretischen Logik* (1928, Berlin: Springer-Verlag)) and the propositional calculus of Whitehead and Russell (*Principia mathematica* (1910, Cambridge: Cambridge University Press)).

In addition, it follows in principle that the system of internal forms is absolutely adequate for the semantic interpretation of the formalizations of classical logic and mathematics, as far as expressions of the propositional calculus are concerned. Although it is somewhat difficult (because of the imperfections of ordinary language) to concisely and conveniently express the internal denial of a proposition of the form "*A* and *B*", for example, the existence in principle of such an internal denial is nevertheless absolutely clear; and it can easily be expressed in ordinary language if we use several definitions, the introduction of which should be unobjectionable.

In accordance with what was said above we shall call internal and external forms, respectively, classical and nonclassical semantic functions of statement variables.

§2. MATRIX FORM OF THE STATEMENT CALCULUS

1. BASIC CONCEPTS AND DEFINITIONS. Let *a, b, c, d,* . . . be statement variables. The range of values of each of these variables has three members—*T* (read "true"), *F* (read "false"), and *N* (read "meaningless")—and no others.

We now introduce the basic functions of statement variables. Each function is defined by a table (matrix), where all the possible combinations of values for the arguments are arranged to the left ot the double lines in a fixed order, and the corresponding values of the functions are arranged to the right.

We introduce, as basic classical functions, formal internal denial $\sim a$ (read "not-*a*") and formal internal logical sum $a \cap b$ (read "*a* and *b*"), defined by the matrices:

1.

a	$\sim a$
T	F
F	T
N	N

2.

a	b	$a \cap b$
T	T	T
T	F	F
F	T	F
F	F	F
T	N	N
N	T	N
F	N	N
N	F	N
N	N	N

As basic nonclassical functions we introduce formal external assertion $\vdash a$ (read "*a* is true") and formal external denial $\urcorner\, a$ (read "*a* is false"), defined by the matrices:

3.

a	$\vdash a$
T	T
F	F
N	F

4.

a	$\urcorner\, a$
T	F
F	T
N	F

The following definitions serve exclusively to simplify the writing of formulas and do not require any explanation (the symbol $\overline{\overline{D}}$ signifies equivalence by definition):

$$\sim\sim a \quad \overline{\overline{D}} \quad \sim(\sim a),$$
$$\sim\vdash a \quad \overline{\overline{D}} \quad \sim(\vdash a),$$
$$\vdash \urcorner a \quad \overline{\overline{D}} \quad \vdash(\urcorner a),$$
$$\urcorner\urcorner \quad \overline{\overline{D}} \quad \urcorner(\urcorner a),$$

etc. for any uninterrupted sequence of a finite number of the symbols $\sim$, $\vdash$, $\urcorner$ and $\downarrow$, which will be introduced below.

We now define several classical functions in terms of classical denial and logical sum:

$$(a \cup b) \quad \overline{\overline{D}} \quad \sim(\sim a \cap \sim b), \qquad (D_1)$$
$$(a \supset b) \quad \overline{\overline{D}} \quad \sim(a \cap \sim b), \qquad (D_2)$$
$$(a \supset\subset b) \quad \overline{\overline{D}} \quad [(a \supset b) \cap (b \supset a)]. \qquad (D_3)$$

The function $a \cup b$—formal internal or classical logical product—is read as "a or b". The function $a \supset b$—formal internal or classical implication—is read as "if a, then b". Using the definitions given above, it is easy to construct matrices for the functions $a \cup b$, $a \supset b$, and $a \supset\subset b$.

Using formal external assertion and formal external denial, we define the following functions:

$$(a \wedge b) \quad \overline{\overline{D}} \quad (\vdash a \cap \vdash b), \qquad (D_4)$$
$$(a \vee b) \quad \overline{\overline{D}} \quad (\vdash a \cup \vdash b), \qquad (D_5)$$
$$(a \rightarrow b) \quad \overline{\overline{D}} \quad (\vdash a \supset \vdash b), \qquad (D_6)$$
$$(a \longleftrightarrow b) \quad \overline{\overline{D}} \quad [(a \rightarrow b) \cap (b \rightarrow a)], \qquad (D_7)$$
$$(a \equiv b) \quad \overline{\overline{D}} \quad [(a \longleftrightarrow b) \cap (\sim a \longleftrightarrow \sim b)], \qquad (D_8)$$
$$\downarrow a \quad \overline{\overline{D}} \quad \sim(\vdash a \cup \urcorner a), \qquad (D_9)$$
$$\bar{a} \quad \overline{\overline{D}} \quad \sim\vdash a.^{8} \qquad (D_{10})$$

The function $a \wedge b$—formal external or nonclassical logical sum—is read as "a is true and b is true". The function $a \vee b$—formal external or nonclassical logical product—is read as "a is true or b is true". The function $a \rightarrow b$—formal external or nonclassical implication—is read as "if a is true, then b is true" or "from the statement a follows the statement b". The function $a \longleftrightarrow b$ is read as "a is of the same strength as b". The function $a \equiv b$ is read as "a is equivalent to b" or "a has the same value as b".

It is interesting to compare the concept of equal strength with the concept of equivalence. If $a \longleftrightarrow b$, then from the truth of one of a or b the truth of the other follows. However, this along does not mean that logically a and b have the same

8 The symbols $\sim$, $\vdash$, $\urcorner$, and $\downarrow$ operate only on the letters and parentheses which stand immediately to their right.

value. If one of them is false, then the other need not be false, but may be meaningless. From

$$a \longleftrightarrow b$$

alone we cannot conclude that

$$\sim a \longleftrightarrow \sim b$$

or that

$$\downarrow a \longleftrightarrow \downarrow b.$$

On the other hand, every statement which follows from a follows from b also, and vice versa. In this particular sense, a and b are of equal strength to one another.

If

$$a \equiv b,$$

then not only does the truth of each of the statements a and b follow from the truth of the other, but in addition the falsehood of one follows from the falsehood of the other and the meaninglessness of one follows from the meaninglessness of the other. The equivalence of two propositions entails their equal strength, but the converse does not in general hold. We note that the matrices of equivalent functions are identical. Thus equivalence in the statement calculus plays the role of "mathematical equality". The function $\downarrow a$ is read as "a is nonsensical" or "a is meaningless". Finally, the function $\bar{a}$ is read as "a is not true".

For the functions $\downarrow a$ and $\bar{a}$, on the basis of their definitions, we obtain the matrices:

5.

a	$\downarrow a$
T	F
F	F
N	T

6.

a	$\bar{a}$
T	F
F	T
N	T

It is easy to construct the matrices for the functions $a \wedge b$, $a \vee b$, $a \rightarrow b$, and $a \equiv b$.

We now define precisely the concept of a formula. This definition is constructed inductively as follows:

1) every statement symbol is a formula,
2) if A is a formula, then $\sim A$, $\vdash A$, and $\urcorner A$ are also formulas,
3) if A and B are formulas, then $A \cap B$ is a formula.

In order to make the writing of formulas less cumbersome, we will use the dot convention.

The symbols $\supset$, $\supset\subset$, $\rightarrow$, $\longleftrightarrow$, $\equiv$, and $\overset{=}{\mathrm{D}}$ are considered to be of one rank, and this rank is higher than that of the symbols $\cap$, $\cup$, $\wedge$, and $\vee$. The latter bind more strongly than the former.

The symbols $\sim$, $\vdash$, $\rceil$, and $\downarrow$ operate only on the letters and parentheses which occur immediately to their right.

The symbol $^{-}$ *always applies to the entire expression over which it stands.*

Thus, the formula

$$a \vee b \rightarrow b \vee a$$

signifies the same as the formula

$$(a \vee b) \rightarrow (b \vee a).$$

The formula

$$a \rightarrow b \,.\cap .\, a \rightarrow \sim b \vee \downarrow b . \rightarrow \sim a \vee \downarrow a$$

signifies the same as the formula

$$\{(a \rightarrow b) \cap [a \rightarrow (\sim b \vee \downarrow b)]\} \rightarrow (\sim a \vee \downarrow a).$$

The definition of the function $a \longleftrightarrow b$ can now be written as

$$a \longleftrightarrow b \,. \overset{=}{\mathrm{D}} .\, a \rightarrow b \,.\cap\, b \rightarrow a,$$

and so on.

A formula is called valid in the matrix statement logic if it has the value T for all possible values of its arguments. Valid formulas are also called tautologies. Proof of validity amounts to verification, which is carried out most systematically and simply by the method of constructing matrices for the given formulas.

A formula which does not have the value T for any values of its arguments is called a contradiction. If A is a contradiction, then $\bar{A}$ is a tautology. And if one of the formulas $\sim A$, $\rceil A$, $\downarrow A$, or $\bar{A}$ is valid, then A is a contradiction.

A formula which contains, besides statement variables, only classical function symbols, is called classical. Let $\phi(a_1, a_2, \ldots, a_n)$ be an arbitrary formula and let its matrix be given. This matrix has 3^n rows. We will call the collection of rows in which none of the arguments has the value N the TF-submatrix for the given formula. Obviously, the TF-submatrix has 2^n rows. The collection of the remaining rows will be called the N-submatrix. The N-submatrix has $3^n - 2^n$ rows.

2. FORMULAS WHICH ARE NOT VALID IN THE STATEMENT CALCULUS.

THEOREM I. No classical formula is valid in the statement calculus.

PROOF. The truth of this assertion is obvious, for every classical formula receives the value N whenever at least one argument receives this value.

THEOREM II. No formula which is a contradiction is valid in the statement calculus.

PROOF. This theorem follows directly from the definitions of valid formula and contradiction in #1 of this section.

The following formulas are examples of contradictions:

$$a \wedge \rceil a,$$
$$a \equiv \rceil a,$$
$$a \longleftrightarrow \bar{a}.$$

THEOREM III. No formula which is constructed from only nonclassical functions can be equivalent to any classical formula.

THEOREM IV. The formula $\downarrow a$ (and consequently, also $\bar{a}$) cannot even be merely equal in strength to any classical formula.

The truth of Theorems III and IV follows directly from the structure of matrices 1–6 in #1 of this section.

3. IMPORTANT FORMULAS WHICH ARE VALID IN THE STATEMENT CALCULUS:

THEOREM V. Every formula which is valid in the classical propositional calculus and which is of the form $A \supset B$,[9] where A and B contain exactly the same variables, remains valid in the statement calculus, if the sign $\supset$ which occurs between A and B is replaced by the sign $\rightarrow$ and the variables are considered to be statement variables.

Similarly, every formula which is valid in the classical propositional calculus and which is of the form $A \supset\subset B$, where A and B contain exactly the same variables, remains valid in the statement calculus, if the sign $\supset\subset$ which occurs between A and B is replaced by the sign $\equiv$ and the variables are considered to be statement variables.

We prove the first part of the theorem. It is evident that if a formula $A \supset B$ is valid in the classical propositional calculus, then in each row in its TF-submatrix the formula $A \rightarrow B$ has the value T.

Now let some variable a_i have the value N. Since the formulas A and B are classical and both, by hypothesis, contain the variable a_i, both receive the value N in this case. But, by the definition of the function $a \rightarrow b$, we have

$$N \rightarrow N = T.$$

Consequently, in each row of its N-submatrix the formula $A \rightarrow B$ also has the value T. But then it always has the value T, and the theorem is proved.

THEOREM VI. The matrix statement calculus contains a part which is isomorphic to the classical matrix propositional calculus, where the formulas of this part of the statement calculus are obtained from the formulas of the classical propositional calculus through the following transformations[10] (the letters c.p.c. below signify the classical propositional calculus and the letters s.c. the statement calculus):

9 Here and below we will suppose that the symbols used in the classical propositional calculus are the same as those used in the classical formulas (in the sense defined above) of the present work.

10 See footnote 9.

1) every propositional variable becomes a statement variable with the same notation,
2) the sign $\sim$ of the c.p.c. becomes the sign $^-$ of the s.c.,
3) the sign $\cap$ of the c.p.c. becomes the sign $\cap$ of the s.c.,
4) the sign $\cup$ of the c.p.c. becomes the sign $\vee$ of the s.c.,
5) the sign $\supset$ of the c.p.c. becomes the sign $\to$ of the s.c.,
6) the sign $\supset\subset$ of the c.p.c. becomes the sign $\leftrightarrow$ of the s.c.

PROOF. It is easy to verify by constructing matrices that the following formulas are tautologies:

$$a \to a \cap a, \tag{1}$$
$$a \cap b \to b \cap a, \tag{2}$$
$$a \to b. \to. a \cap c \to b \cap c, \tag{3}$$
$$a \to b. \cap. b \to c. \to .a \to c, \tag{4}$$
$$b \to .a \to b, \tag{5}$$
$$a \cap. a \to b. \to b, \tag{6}$$
$$a \to a \vee b, \tag{7}$$
$$a \vee b \to b \vee a, \tag{8}$$
$$a \to c. \cap .b \to c. \to a \vee b \to c, \tag{9}$$
$$\bar{a} \to .a \to b, \tag{10}$$
$$a \to b. \cap .a \to \bar{b}. \to \bar{a} \tag{11}$$
$$a \vee \bar{a} \tag{12}$$

The system of formulas (1)–(12) is an isomorphic image of the following system of formulas of the classical propositional calculus:

$$a \supset a \cap a,$$
$$a \cap b \supset b \cap a,$$
$$a \supset b. \supset .a \cap c \supset b \cap c,$$
$$a \supset b. \cap .b \supset c. \supset .a \supset c,$$
$$b \supset .a \supset b,$$
$$a \cap .a \supset b. \supset b,$$
$$a \supset a \cup b,$$
$$a \cup b \supset b \cup a,$$
$$a \supset c. \cap .b \supset c. \supset .a \cup b \supset c,$$
$$\sim a \supset .a \supset b,$$
$$a \supset b. \cap .a \supset \sim b. \supset \sim a,$$
$$a \cup \sim a.$$

But this system, as is well known,[11] can be used as the formal axiom system for classical propositional logic, if the following are introduced:

11 A. Heyting, 'Die formalen Regeln der intuitionistischen Logik', *Sitzungsberichte der Preussischen Akademie der Wissenschaften, physikalisch-mathematische Klasse*, (1930), 42–56; A. Kolmogoroff, 'Zur Deutung der intuitionistischen Logik', *Mathematische Zeitschrift*, 35 (1932), 58–65.

1) principle of detachment:
if a and $a \supset b$ are valid formulas, then b is a valid formula,
2) rule of conjunction:
if a and b are valid formulas, then $a \cap b$ is a valid formula.
3) principle of substitution in the usual form.

Now we can see from the matrix of the function $a \to b$ that in the matrix statement calculus the principle of detachment holds in the form:

if a and $a \to b$ are valid formulas, then b is a valid formula.

Further, from the matrix of the function $a \wedge b$, we can see that in the matric statement calculus the rule

if a and b are valid formulas, then $a \wedge b$ is a valid formula

also holds.

Finally, it is obvious that the principle of substitution holds for the matrix statement calculus. These claims establish Theorem VI.

The isomorphic image of classical propositional logic whose existence has just been established is called the K_1-system. It would be easy to show that the statement calculus contains another isomorphic image of propositional logic, obtained from the K_1-system by replacing the symbol $\cap$ with the symbol $\wedge$ and the symbol $\leftrightarrow$ with the symbol $\equiv$. This second isomorphic image of the classical propositional calculus is called the K_2-system.[12]

Theorems V and VI permit us easily to specify a large class of formulas which are valid in the statement calculus. Thus, the following formulas serve to illustrate Theorem V:

$$a \equiv \sim\sim a, \tag{13}$$
$$\sim (a \cap b) \equiv \sim a \cup \sim b, \tag{14}$$
$$\sim (a \cup b) \equiv \sim a \cap \sim b, \tag{15}$$
$$\sim (a \supset b) \equiv a \cap \sim b, \tag{16}$$
$$a \supset \sim a. \to \sim a. \tag{17}$$

However, we should not overestimate the operative strength of these formulas; it is necessary to remember that classical formulas stand to the left and right of the signs $\equiv$ and $\to$ in these formulas, and to keep Theorem I in #2 of this section in mind.

We now consider a further series of important formulas of the statement calculus. First we display the fundamental formulas signifying the connections between classical and nonclassical functions:

$$a \leftrightarrow \vdash a, \tag{18}$$
$$\sim a \leftrightarrow \urcorner a, \tag{19}$$
$$a \cap b \leftrightarrow a \wedge b, \tag{20}$$
$$a \cup b \to a \vee b, \tag{21}$$
$$a \supset b. \to .a \to b. \tag{22}$$

It is very important to stress that the last two formulas contain only one-directional nonclassical implication, while the first three contain the equal-strength connective.

12 We note that the symbol $\cap$ may be replaced in the K_1-system by the symbol $\wedge$ without simultaneously replacing the symbol $\leftrightarrow$ by the symbol $\equiv$; this, however, is not of independent interest.

Two formulas which connect meaninglessness with classical and nonclassical denials are:

$$\downarrow a \equiv \downarrow \sim a, \qquad (23)$$
$$\downarrow a \rightarrow \rceil \rceil a. \qquad (24)$$

Formula (25) shows that the external assertion of a meaningless statement is false:

$$\downarrow a \rightarrow \rceil \vdash a. \qquad (25)$$

Some interesting formulas are:

$$\overline{\sim (a \cup \sim a)}, \qquad (26)$$
$$\rceil \rceil (a \cup \sim a), \qquad (27)$$
$$\rceil \downarrow (a \vee \rceil a), \qquad (28)$$
$$\downarrow (a \cup \sim a) \equiv \rceil (a \vee \rceil a), \qquad (29)$$
$$\downarrow (a \cup \sim a) \equiv \downarrow a. \qquad (30)$$

From formula (26) we can see that the classical denial of the classical form "tertium non datur" is always false or meaningless. Formula (27) asserts that the nonclassical denial of the classical form "tertium non datur" is always false.

Formula (28) asserts that the nonclassical form "tertium non datur" cannot be meaningless, and, particularly, that it is always false that it is meaningless.

Formula (29) shows that for "tertium non datur" the meaninglessness of the classical form is equivalent to the falsehood of the nonclassical. Finally, formula (30) asserts that the classical form "tertium non datur" is meaningless if and only if the given statement is meaningless.

Now we note formulas which are especially important for the analysis of paradoxes:

$$a \equiv \sim a . \equiv \downarrow a, \qquad (31)$$
$$a \longleftrightarrow \sim a . \equiv \downarrow a, \qquad (32)$$
$$\sim\downarrow a \rightarrow . a \longleftrightarrow \sim a : \equiv \downarrow a, \qquad (33)$$
$$a \cup \sim a \rightarrow . a \longleftrightarrow \sim a : \equiv \downarrow a, \qquad (34)$$
$$a \longleftrightarrow \rceil a . \equiv \downarrow a, \qquad (35)$$
$$\vdash a \equiv \rceil a . \equiv \downarrow a. \qquad (36)$$

However, the formula

$$a \equiv \rceil a . \equiv \downarrow a$$

does not hold. If a is meaningless, then $\downarrow a$ is true, but $a \equiv \rceil a$ is always a false formula.

Consider further, for comparison, the formulas

$$a \equiv \downarrow a . \equiv \rceil a, \qquad (37)$$
$$a \equiv \downarrow a . \longleftrightarrow \sim a. \qquad (38)$$

The following formulas are important:

$$\sim a \rightarrow . a \rightarrow b, \qquad (39)$$
$$\rceil a \rightarrow . a \rightarrow b, \qquad (40)$$
$$\downarrow a \rightarrow . a \rightarrow b, \qquad (41)$$
$$a \equiv b . \rightarrow . \sim a \equiv \sim b, \qquad (42)$$
$$a \equiv b . \rightarrow . \downarrow a \equiv \downarrow b. \qquad (43)$$

II

THE RESTRICTED FUNCTIONAL CALCULUS

§1. BASIC CONCEPTS, NOTATION, AND DEFINITIONS

The variables of the functional calculus are divided into three groups:

1) statement variables: $a, b, c, \ldots,$
2) individual variables: $x, y, z, \ldots,$
3) functional variables of any finite number of individual variables:

$$f(\ ), g(\ ), \ldots, \phi(\ ), \psi(\ ), \ldots$$

To these three groups of variables correspond three groups of constants, the notation for which is specially introduced as needed.

The expression $f(x)$ is read: "x has the property f". The expression $f(x, y)$ is read as: "x stands in the relation f to y". The symbol (x), the basic quantifier, is called the universal quantifier. The expression $(x)f(x)$ is read as: "every x has the property f".

The concept "formula" is defined inductively by the following rules (we will sometimes call a formula a "statement"):

1) Each statement symbol is a formula.
2) A function symbol whose argument places have been filled with individual constants or individual variables is a formula.
3) If A is a formula and A contains a free variable x (depends on x), then $(x)A$ is is a formula.
4) If A is a formula, then $\sim A$, $\rceil A$, and $\vdash A$ are formulas.
5) If A is a formula and B is a formula, then $A \cap B$ is a formula.
6) If any part of a formula is covered by a universal quantifier binding some variable, then no other universal quantifier which binds the same variable can cover that part of the formula.

Further, in the functional calculus definitions (D_1)–(D_{10}) are introduced, from I, §2, #1. Hence, if A is a formula, then $\downarrow A$ and $\bar{A}$ are both formulas, and if A and B are formulas, then $A \cup B$, $A \supset B$, $A \supset\subset B$, $A \wedge B$, $A \vee B$, $A \rightarrow B$, $A \longleftrightarrow B$, and $A \equiv B$ are also formulas.

Using the basic quantifier (x), we now introduce three new ones: (εx), $\exists x$, and $\forall x$. These quantifiers are defined as:

$$(\varepsilon x) f(x) \;\overline{\overline{\mathrm{D}}}\; \sim (x) \sim f(x), \qquad (\mathrm{D}_{11})$$
$$\exists x f(x) \;\overline{\overline{\mathrm{D}}}\; (\varepsilon x) \vdash f(x), \qquad (\mathrm{D}_{12})$$
$$\forall x f(x) \;\overline{\overline{\mathrm{D}}}\; (x) \vdash f(x). \qquad (\mathrm{D}_{13})$$

Thus if A is a formula which contains the free variable x, then $(\varepsilon x) A$, $\exists x A$ and $\forall x A$ are formulas.

$(\varepsilon x) f(x)$ is read as: "there exists at least one x with the property f",
$\exists x f(x)$ is read as: "for at least one x, the statement $f(x)$ is true",
$\forall x f(x)$ is read as: "the statement $f(x)$ is true for all x".

Because of their properties specified in the axiom system below, the quantifiers (x) and (εx) are called, respectively, the classical universal quantifier and the classical existential quantifier.

The quantifiers $\forall x$ and $\exists x$ are called, respectively, the nonclassical universal quantifier and the nonclassical existential quantifier.

Let us now agree that, in the grouping of formulas with dots, the signs $\supset\subset$, $\supset$, $\rightarrow$, $\longleftrightarrow$, $\equiv$, $\overline{\overline{\mathrm{D}}}$, $\cup$, $\cap$, $\vee$, and $\wedge$ have priority over quantifiers; and the signs $\sim$, $\vdash$, $\rceil$, and $\downarrow$, when placed before quantifiers, cover the whole formula consisting of the quantifier together with its scope. In the remaining cases the dot rules have priority.

Thus, the statement
$$(x) . f(x) \rightarrow g(x)$$
signifies the same as
$$(x)(f(x) \rightarrow g(x)),$$
the formula
$$(x) . f(x) \cap g(x) . \rightarrow (x) h(x)$$
signifies the same as
$$(x)(f(x) \cap g(x)) \rightarrow (x) h(x),$$
and the formula
$$\rceil\, (x) . f(x) \rightarrow g(x)$$
signifies the same as
$$\rceil\, ((x)(f(x) \rightarrow g(x))).$$
Finally, instead of $\overline{(x) f(x)}$, $\overline{(\varepsilon x) f(x)}$, $\overline{\forall x f(x)}$, and $\overline{\exists x f(x)}$ we shall write, respectively,
$$(\bar{x}) f(x),\ (\overline{\varepsilon x}) f(x),\ \bar{\forall} x f(x),\ \bar{\exists} x f(x).$$

§2. AXIOMS OF THE RESTRICTED FUNCTIONAL CALCULUS

We use the following three groups of axioms:

I. Every tautological formula of the statement calculus is a valid formula.

II:

II_1) $(x) f(x) \rightarrow f(y)$.[13]

II_2) $f(y) \rightarrow \exists x f(x)$.

II_3) $\downarrow (x) f(x) \rightarrow \exists x \downarrow f(x)$.

II_4) $\exists x \downarrow f(x) \rightarrow \downarrow (x) f(x)$.

13 The variable y does not occur bound in $f(x)$.

III:

III_1) All the axioms of II are valid formulas.

III_2) If A and B are valid formulas, then $A \cap B$ is a valid formula.

III_3) If A and $A \rightarrow B$ are valid formulas, then B is a valid formula (principle of external detachment). Schematically:

$$\frac{A, \quad A \rightarrow B,}{B.}$$

III_4) Principle of substitution: in a valid formula, the following substitutions may be made to obtain another true formula:

1) a single formula may be substituted for every occurrence of some statement variable in the given formula,
2) a single formula which depends on the variables $x, y, \ldots, u$ (and possibly on other variables) may be substituted for every occurrence of some function variable with arguments $x, y, \ldots, u$.
3) an individual variable may be everywhere replaced by another individual variable or an individual constant which names an individual belonging to the range of values of the given variable.

It goes without saying that

1) the principle of substitution applies only to free variables and
2) in place of a variable which stands in the scope of some quantifier, it is not permissible to substitute an expression which depends on the variable which this quantifier binds.

In the restricted functional calculus, we understand individuals to be objects belonging to a definite, limited, and previously defined range.

III_5) Schema for quantifiers:

1) If $B(x)$ is an expression depending on x, A is an expression which does not depend on x, and $A \rightarrow B(x)$ is a valid formula, then $A \rightarrow (x) B(x)$ is also a valid formula,
2) if $B(x)$ is an expression depending on x, A is an expression not depending on x, and $B(x) \rightarrow A$ is a valid formula, then $\exists x B(x) \rightarrow A$ is also a valid formula.

§3. SOME VALID RULES AND FORMULAS OF THE RESTRICTED FUNCTIONAL CALCULUS

THEOREM VII. In the restricted functional calculus of the system under consideration the following rule holds: if A and $A \supset B$ are valid formulas, then B is a valid formula (principle of internal detachment). Schematically:

$$\frac{A. \quad A \supset B,}{B.}$$

PROOF. Let A and $A \supset B$ be valid formulas; applying the principle of external detachment III_3) to the formula $A \supset B$ and the valid formula

$$A \supset B. \rightarrow .A \rightarrow B$$

(see formula (28), I, §2, #3), it follows that

$$\frac{A \supset B, \quad A \supset B. \rightarrow .A \rightarrow B,}{A \rightarrow B,}$$

i.e., $A \rightarrow B$ is a valid formula. Now, by virtue of the assumption of the validity of the formula A, again applying the principle of external detachment, we have

$$\frac{A, \quad A \rightarrow B,}{B,}$$

i.e., B is a valid formula, and the theorem is proved.

THEOREM VIII. The restricted functional calculus of the system under consideration contains a part which is isomorphic to the classical restricted functional calculus.

PROOF. The restricted functional calculus contains the formulas (1)–(12), I, §2, #3. Adding to these formulas the axioms of group II and groups III, it is evident that we obtain in our restricted functional calculus an isomorphic image of the classical functional calculus, where the universal quantifier of the classical calculus corresponds to the quantifier (x) of our calculus, and the existential quantifier of the classical calculus corresponds to the symbol $\exists x$ of our calculus. The theorem is proved.

The isomorphic image of the classical restricted functional calculus whose existence has just been proved will again be called the K_1-system. Now it is easy to find a whole class of formulas which are valid in the restricted functional calculus of the system under consideration.

We note:

1) The principle of generalization: let the formula $A(x)$ which contains the free variable x be valid; then the formula

$$(x)A(x)$$

is also valid.

2) The formulas:

$$f(n) \rightarrow \exists x f(x) \tag{44}$$

(here n is an individual constant which names an individual belonging to the range of values of the variable x), and

$$(x).f(x) \rightarrow g(x). \rightarrow .(x)f(x) \rightarrow (x)g(x). \tag{45}$$

We note also some formulas which are not contained in the K_1-system:

$$(\varepsilon x)f(x) \rightarrow \exists x f(x), \tag{46}$$

$$\sim (\varepsilon x)f(x) \equiv (x) \sim f(x), \tag{47}$$

$$\sim (x)f(x) \equiv (\varepsilon x) \sim f(x), \tag{48}$$

$$\exists x f(x) \cap \sim \downarrow (\varepsilon x)f(x) \rightarrow (\varepsilon x)f(x). \tag{49}$$

Finally, note that the following holds:

THEOREM IX. If $f(x) \equiv g(x)$ is a valid formula, then $(x)f(x) \equiv (x)g(x)$ is also a valid formula.

III.

THE EXTENDED FUNCTIONAL CALCULUS AND THE ANALYSIS OF PARADOXES

§1. THE EXTENDED FUNCTIONAL CALCULUS

The analysis of the paradoxes of the classical system through the formal apparatus of our calculus depends on the possibility of constructing in our system each formula examined in the classical system. It is evident that the apparatus of the restricted functional calculus is inadequate for this. Therefore some extension of the calculus is necessary. Such an extension is the aim of this section.

First, using only some of the elements of the system considered above, we construct a new system which we call system Σ_0. We begin by constructing the statement calculus of the system Σ_0, introducing just two functions of statement variables, $\sim a$ and $a \cap b$, defined as in I, §2. We include the definitions (D_1)–(D_3). In other words, we introduce classical functions, but no nonclassical ones. Obviously, the concepts of formula and statement are now correspondingly narrower than those in I, §2. The definitions of the concepts tautology and contradiction remain as earlier.

It is easy to see that in the statement calculus of system Σ_0 there are no valid formulas.

Now we turn to the construction of the restricted functional calculus of Σ_0. We will proceed as we did in II, §1, right up to the definition of the concept of formula. The latter is defined through the following rules:

1) each statement symbol (in the sense of Σ_0) is a formula,
2) each function symbol whose argument places have been filled with individual constant or individual variables is a formula,
3) if A is a formula containing the free variable x, then $(x)A$ is a formula,
4) if A is a formula, then $\sim A$ is a formula,
5) if A and B are formulas, then $A \cap B$ is a formula,
6) if some part of a formula is covered by a universal quantifier binding some variable, then this part of the formula cannot be covered by another universal quantifier which binds the same variable.

Further we introduce, once again, definitions (D_1)–(D_3) and (D_{11}). Thus if A and B are formulas, $A \cup B$, $A \supset B$, and $A \supset\subset B$ are also formulas; and if A is a formula which contains a free variable x, then $(\varepsilon x)A$ is also a formula.

The rules for writing formulas remain the same.

Of the axioms we keep only I, III_2, and III_4, i.e., only those which concern classical formulas.

It is evident that the restricted functional calculus of Σ_0 contains no valid formulas.

Now we extend the functional calculus of Σ_0, to include under the concept of individuals not only the original objects but also all possible functions and statements of system Σ_0. Thus in axiom III_4 the concept of individual is interpreted in this extended sense. Thus functions of functions and functions of statements are introduced for consideration. Nevertheless, the argument place of each function has a fixed range of individuals. Examples of such ranges are the collection of all statements in the sense of Σ_0 or, say, the collection of functions in the sense of Σ_0. We will call the system thus obtained the full system Σ_0.

It is completely evident that in the full system Σ_0 we have exactly the same collection of formulas as are considered in the extended functional calculus of Hilbert and Ackermann, unrestricted by the theory of types,[14] so that if the range of objects is the same in both systems then the variables are the same in both systems.

It is also evident that the full system Σ_0 contains no valid formulas. Formulas may only be examined in the system.

We now expand system Σ_0 as follows:

1) We introduce nonclassical assertion and denial as functions of statement variables, with the same properties as in I, §2, and we also introduce all of the definitions (D_4)–(D_{10}) and (D_{12}) and (D_{13}).
2) We correspondingly expand the concepts of statement and function. For this, of course, we expand also the concept of formula, but with one restriction which it is necessary to emphasize: except for objects, only functions and statements in the sense of the full system Σ_0 fall under the concept of individual. In other words, the range of individuals remains identical with that of the full system Σ_0.

In axioms I, III_2, and III_4 we will understand the words "statement", "function" and "formula" in the new broadened sense. However, we will interpret the concept of individual in part 3 of axiom III_4 in accordance with the aforementioned restriction.

3) We note that the axioms of groups II, and also axioms III_1, III_3 and III_5 use the words "function", "statement", and "formula" in the new sense, corresponding to part 2 above.

We call the system now obtained system Σ. It is evident that in system Σ we can, and must, distinguish function and statement variables in the sense of system Σ_0 from function and statement variables in the broader sense of system Σ. We will call functions (respectively, statements) in the sense of system Σ_0 simpy functions (respectively, statements) of classical logic, which is evidently entirely legitimate in view of the correspondence between system Σ_0 and the extended functional calculus of Hilbert and Ackermann. For function variables of classical logic we introduce the notation:

$$f_c(\ ),\ g_c(\ ),\ \ldots,\ \phi_c(\ ),\ \psi_c(\ ),\ \ldots$$

14 See Hilbert and Ackermann (footnote 7), 82–115.

For statement variables of classical logic we introduce the notation:

$$a_c,\ b_c,\ c_c,\ \ldots$$

For function and statement variables in the sense of system Σ we retain the notation of II, §1.

It is easy to prove that in system Σ there can be no expression of classical logic which is of the same strength as $\downarrow a_c$. We first note that because of the claims in Theorem IV of I, §2, #2, it is sufficient to show that each expression of classical logic of the form $(b_c)F(a_c, b_c)$ is meaningless if $\downarrow a_c$ is true. But this is clear, since if $\downarrow a_c$ is true then a_c is meaningless; and thus $F(a_c, b_c)$ is certainly meaningless as well, i.e., the formula

$$\downarrow F(a_c, b_c)$$

holds. Consequently, by axiom II_2,

$$\exists b_c \downarrow F(a_c, b_c)$$

and further, by axiom II_4 we have:

$$\downarrow (b_c) F(a_c, b_c),$$

which required proof.

All this reasoning, of course, presupposes the consistency of system Σ. The consistency of the latter remains for the time being unproved, but all the author's attempts to obtain a contradiction in it were vain, so that the empirical grounds for holding system Σ to be consistent appear to be important enough.[15]

And system Σ is the extension of the functional calculus necessary for the analysis of the paradoxes. So we turn to the analysis.

§2. ANALYSIS OF THE PARADOXES OF CLASSICAL MATHEMATICAL LOGIC

1. SOME GENERAL REMARKS. The paradoxes of the classical extended functional calculus can be divided into two groups. The paradoxes of the first group have a purely logical nature and do not require for their construction any assumptions lying outside the range of purely logical formulas. An example of a paradox of this kind is Russell's paradox. The paradoxes of the second group cannot be constructed without augmenting the apparatus of the logical calculus with well-known formulas containing symbols for particular individuals—functions or propositions. An example of a paradox of this second group is Weyl's "heterologisch" paradox.[16]

15 [Bochvar worked on proofs of the consistency of system Σ in the works cited in footnote 3 above.]

16 On the difference between these two groups of paradoxes see: Hilbert and Ackermann (footnote 7), 115; F. Ramsey, 'The foundations of mathematics', *Proceedings, London Mathematical Society*, (2) 25 (1926), 338–384; R. Carnap, *Abriss der Logistik* (Vienna, 1929), 21; R. Carnap, 'Die Antinomien und die Unvollständigkeit der Mathematik', *Monatschefte für Mathematik und Physik*, 41 (1934), 263–284 [in English in *Logical syntax of language* (New York and London, 1937)].

While in the case of the paradoxes of the first type the apparatus of system Σ allows us to show directly the meaninglessness of certain statements, in the case of paradoxes of the second type the results of analysis are based upon assumptions of the sort mentioned, as the construction of these paradoxes in the classical system is based upon them.

The analysis of Russell's and Weyl's paradoxes is presented below. In this section functions of one variable are sometimes called properties. And we will, for brevity, sometimes write simply ϕ instead of the symbol $\phi(\ )$.

2. ANALYSIS OF RUSSELL'S PARADOX. In the extended functional calculus of Hilbert and Ackermann, Russell's paradox is obtained by considering the function

$$\phi(\phi),$$

which expresses the second-order property of applying to itself. We define

$$\mathrm{Pd}\,(\phi)\ \overline{\overline{\mathrm{D}}}\ \phi(\phi).$$

In virtue of the validity in classical logic of the formula

$$a \supset\subset a$$

we may write

$$\phi(\phi) \supset\subset \phi(\phi),$$

or, using the definition of the function Pd,

$$\phi(\phi) \supset\subset \mathrm{Pd}\,(\phi).$$

The function $\sim$Pd belongs to the range of values of the variable ϕ. Substituting $\sim$Pd for ϕ in the last formula, we obtain:

$$\sim\mathrm{Pd}(\sim\mathrm{Pd}) \supset\subset \mathrm{Pd}(\sim\mathrm{Pd}).$$

And this is Russell's contradiction.

Now we shall see how things stand in system Σ. We consider the function

$$\phi_c(\phi_c)$$

and we define

$$\mathrm{Pd}(\phi_c)\ \overline{\overline{\mathrm{D}}}\ \phi_c\,(\phi_c).$$

We note that the range of values of the variable ϕ_c coincides with the range of the variable ϕ, introduced earlier for the examination of Russell's paradox in the classical extended functional calculus. The formula

$$a \supset\subset a$$

however, cannot be used in system Σ since here it is not valid.

On the other hand we have the valid formula

$$a \equiv a.$$ [17]

We substitute $\phi_c(\phi_c)$ for a in this formula and obtain

$$\phi_c(\phi_c) \equiv \phi_c(\phi_c).$$

Now, using the definition of the function Pd, we may write

$$\mathrm{Pd}(\phi_c) \equiv \phi_c(\phi_c).$$

The function $\sim$Pd belongs to the range of values of the variable ϕ_c. Substituting $\sim$Pd for ϕ_c in the last formula, we obtain:

$$\mathrm{Pd}(\sim\mathrm{Pd}) \equiv \sim\mathrm{Pd}(\sim\mathrm{Pd}). \qquad (\alpha)$$

Because of the validity in system Σ of the formula

$$a \equiv \sim a.\ \equiv \downarrow a,$$ [18]

it follows that

$$\mathrm{Pd}(\sim\mathrm{Pd}) \equiv \sim\mathrm{Pd}(\sim\mathrm{Pd}).\ \equiv \downarrow \mathrm{Pd}(\sim\mathrm{Pd})$$

and, from (α),

$$\downarrow\mathrm{Pd}(\sim\mathrm{Pd}).$$

Further, from the validity of the formula

$$\downarrow a \equiv \downarrow \sim a$$ [19]

it follows that

$$\downarrow \sim \mathrm{Pd}(\sim\mathrm{Pd}).$$

Thus, the statement Pd($\sim$Pd) is meaningless, as is its internal denial. The external denial of the statement Pd($\sim$Pd) is false; and so is its external assertion.

17 See I, §2, #3, Theorem V.
18 See I, §2, #3, formula (31).
19 See I, §2, #3, formula (23).

[3.] ANALYSIS OF WEYL'S PARADOX. We first present the formal construction of Weyl's paradox in the classical extended functional calculus. That the symbol z is heterological is expressed by the function $H(z)$, defined as follows:

$$H(z) \overline{\overline{\mathrm{D}}} . (\varepsilon\phi) . R(z, \phi) \cap \sim \phi(z).$$ [20]

Here $R(z, \phi)$ is read as: "z designates ϕ". The range of values of the variable z is the collection of symbols designating properties, and the range of values of ϕ is the collection of those properties. We take it as an axiom that the symbol "H" designates only the function H. Symbolically this axiom may be expressed by the formulas:

1) $R(\text{“}H\text{”}, H)$,
2) $R(\text{“}H\text{”}, \phi) \supset \phi = H$.

Identity is defined in classical mathematical logic in accordance with the formula

$$x = y \overline{\overline{\mathrm{D}}} . (f) . f(x) \supset f(y).$$ [21]

Therefore formula 2 may be rewritten as:

2) $R(\text{“}H\text{”}, \phi) \supset . (f) . f(\phi) \supset f(H)$.

Now from the definition of the function H we have:

$$H(\text{“}H\text{”}) \supset . (\varepsilon\phi) . R(\text{“}H\text{”}, \phi) \cap \sim \phi(\text{“}H\text{”}). \qquad (\alpha)$$

From 2 and the valid formula

$$(f) . f(\phi) \supset f(H) . \supset . g(\phi) \supset g(H)$$

we obtain

$$R(\text{“}H\text{”}, \phi) \supset . g(\phi) \supset g(H).$$

Substituting the function $\sim\phi(\text{“}H\text{”})$ for $g(\phi)$ in this formula we obtain

$$R(\text{“}H\text{”}, \phi) \supset . \sim \phi(\text{“}H\text{”}) \supset \sim H(\text{“}H\text{”}).$$

Further, from this we have

$$R(\text{“}H\text{”}, \phi) \cap \sim \phi(\text{“}H\text{”}) \supset . \sim \phi(\text{“}H\text{”}) \cap . \sim \phi(\text{“}H\text{”}) \supset \sim H(\text{“}H\text{”}).$$

Since the formula

20 See Ramsey (footnote 16).
21 See, for example, Carnap, *Abriss der Logistik* (footnote 16), 15; also Hilbert and Ackermann (footnote 7), 83.

$$\sim \phi(\text{“}H\text{”}) \cap . \sim \phi(\text{“}H\text{”}) \supset \sim H(\text{“}H\text{”}) . \supset \sim H(\text{“}H\text{”})$$

holds, we have

$$R(\text{“}H\text{”}, \phi) \cap \sim \phi(\text{“}H\text{”}) \supset \sim H(\text{“}H\text{”}).$$

Applying a well-known rule of Hilbert and Ackermann's functional calculus to the last formula, we may write

$$(\varepsilon\phi) . R(\text{“}H\text{”}, \phi) \cap \sim \phi(\text{“}H\text{”}) . \supset \sim H(\text{“}H\text{”}).$$

Taking (α) into consideration, we have

$$H(\text{“}H\text{”}) \supset \sim H(\text{“}H\text{”}). \qquad (A)$$

On the other hand, from the valid formula

$$f(n) \supset (\varepsilon x) f(x),$$

where n is an individual constant which names an individual belonging to the range of values of the variable x, we may write:

$$R(\text{“}H\text{”}, H) \cap \sim H(\text{“}H\text{”}) \supset . (\varepsilon\phi) . R(\text{“}H\text{”}, \phi) \cap \sim \phi(\text{“}H\text{”}),$$

or, by the definition of the function H,

$$R(\text{“}H\text{”}, H) \cap \sim H(\text{“}H\text{”}) \supset H(\text{“}H\text{”}).$$

But since $R(\text{“}H\text{”}, H)$ is adopted as an axiom, it follows that

$$\sim H(\text{“}H\text{”}) \supset H(\text{“}H\text{”}). \qquad (B)$$

Formulas (A) and (B) together yield Weyl's paradox:

$$H(\text{“}H\text{”}) \supset\subset \sim H(\text{“}H\text{”}).$$

We will investigate now what may be obtained in system Σ. Since the calculations here are considerably long, for the sake of brevity we shall not in general refer to the formulas of the statement calculus used in the proof. It is easy to see in each case which formula is used, and it is also easy to verify this formula, perhaps with the help of its corresponding matrix.

First of all we define in system Σ the function $H(z)$:

$$H(z) \overline{\overline{\mathrm{D}}} . (\varepsilon\phi_c) . R(z, \phi_c) \cap \sim \phi_c(z).$$

$R(z, \phi_c)$ is read as: "z designates ϕ_c".

The range of values of the variable z is the collection of symbols of classical logic which designate properties, and the range of values of ϕ_c is the collection of those properties which are considered in classical logic. Thus the ranges of values of the variables z and ϕ_c are the same as the ranges of z and ϕ used to construct Weyl's paradox in classical logic.

Formulas 1 and 2 correspond now to the formulas

1′) $R(\text{“}H\text{”}, H)$,
2′) $R(\text{“}H\text{”}, \phi_c) \rightarrow .(f_c).f_c(\phi_c) \rightarrow f_c(H)$.[22]

By the definition of the function H, we have

$$H(\text{“}H\text{”}) \rightarrow .(\varepsilon\phi_c).R(\text{“}H\text{”}, \phi_c) \cap \sim \phi_c(\text{“}H\text{”}). \qquad \text{(a)}$$

From 2′ with the help of axiom II_1 (II, §2), follows

$$R(\text{“}H\text{”}, \phi_c) \rightarrow .g_c(\phi_c) \rightarrow g_c(H). \qquad \text{(b)}$$

Substituting $\sim \phi_c(\text{“}H\text{”})$ for $g_c(\phi_c)$ in this formula, we obtain

$$R(\text{“}H\text{”}, \phi_c) \rightarrow . \sim \phi_c(\text{“}H\text{”}) \rightarrow \sim H(\text{“}H\text{”}). \qquad \text{(c)}$$

From this we obtain, further,

$$R(\text{“}H\text{”}, \phi_c) \cap \sim \phi_c(\text{“}H\text{”}) \rightarrow . \sim \phi_c(\text{“}H\text{”}) \cap . \sim \phi_c(\text{“}H\text{”}) \rightarrow \sim H(\text{“}H\text{”}). \quad \text{(d)}$$

Since the formula

$$\sim \phi_c(\text{“}H\text{”}) \cap . \sim \phi_c(\text{“}H\text{”}) \rightarrow \sim H(\text{“}H\text{”}). \rightarrow \sim H(\text{“}H\text{”})$$

is valid, we obtain

$$R(\text{“}H\text{”}, \phi_c) \cap \sim \phi_c(\text{“}H\text{”}) \rightarrow \sim H(\text{“}H\text{”}). \qquad \text{(e)}$$

The application of axiom III_5 from II, §2 (schema for the quantifiers) to formula (e) yields

$$\exists\phi_c.R(\text{“}H\text{”}, \phi_c) \cap \sim \phi_c(\text{“}H\text{”}). \rightarrow \sim H(\text{“}H\text{”}). \qquad \text{(f)}$$

Using formula (46), II, §3, we obtain now

$$(\varepsilon\phi_c).R(\text{“}H\text{”}, \phi_c) \cap \sim \phi_c(\text{“}H\text{”}). \rightarrow \sim H(\text{“}H\text{”}), \qquad \text{(g)}$$

or, by the definition of the function H,

22 This condition is even weaker than the identity of the functions designated by the symbol “H”.

$$H(\text{“}H\text{”}) \rightarrow \sim H(\text{“}H\text{”}). \qquad \text{(A′)}$$

On the other hand, on the basis of formula (44), II, §3, we have

$$R(\text{“}H\text{”}, H) \cap \sim H(\text{“}H\text{”}) \rightarrow .\exists\phi_c.R(\text{“}H\text{”}, \phi_c) \cap \sim \phi_c(\text{“}H\text{”}). \qquad \text{(h)}$$

But by formula (49), II, §3 and from the definition of the function H:

$$\sim \downarrow H(\text{“}H\text{”}) \cap .\exists\phi_c.R(\text{“}H\text{”}, \phi_c) \cap \sim \phi_c(\text{“}H\text{”}). \rightarrow H(\text{“}H\text{”}). \qquad \text{(i)}$$

From formula (i) follows

$$\exists\phi_c.R(\text{“}H\text{”}, \phi_c) \cap \sim \phi_c(\text{“}H\text{”}). \rightarrow . \sim \downarrow H(\text{“}H\text{”}) \rightarrow H(\text{“}H\text{”}). \qquad \text{(k)}$$

From formulas (h) and (k) we have

$$R(\text{“}H\text{”}, H) \cap \sim H(\text{“}H\text{”}) \rightarrow . \sim \downarrow H(\text{“}H\text{”}) \rightarrow H(\text{“}H\text{”}), \qquad \text{(l)}$$

or

$$R(\text{“}H\text{”}, H) \rightarrow : \sim H(\text{“}H\text{”}) \rightarrow . \sim \downarrow H(\text{“}H\text{”}) \rightarrow H(\text{“}H\text{”}). \qquad \text{(m)}$$

Since the formula $R(\text{“}H\text{”}, H)$ has been adopted as an axiom, we obtain

$$\sim H(\text{“}H\text{”}) \rightarrow . \sim \downarrow H(\text{“}H\text{”}) \rightarrow H(\text{“}H\text{”}),$$

or, changing the positions of the antecedents,

$$\sim \downarrow H(\text{“}H\text{”}) \rightarrow . \sim H(\text{“}H\text{”}) \rightarrow H(\text{“}H\text{”}). \qquad \text{(B′)}$$

On the other hand, by the validity of formula (A′) the formula

$$\sim \downarrow H(\text{“}H\text{”}) \rightarrow . H(\text{“}H\text{”}) \rightarrow \sim H(\text{“}H\text{”}) \qquad \text{(A″)}$$

is also valid. From (A″) and (B′) we have

$$\sim \downarrow H(\text{“}H\text{”}) \rightarrow . H(\text{“}H\text{”}) \rightarrow \sim H(\text{“}H\text{”}),$$

from which, by formula (33), I, §2, #3, the formula

$$\downarrow H(\text{“}H\text{”})$$

follows, and, further, by formula (23), I, §2, #3,

$$\downarrow \sim H(\text{“}H\text{”}).$$

Resume

[This resume was published in English in *Matematicheskii sbornik*; it appeared with the article.]

In the present paper a three-valued logical calculus is investigated in which senselessness is introduced as the third possible truth-value of an outsaying[23] and which allows an exact analysis of the contradictions of the classical mathematical logic. This analysis proceeds in the form of a formal proof and results in each investigated case in a formula stating that a quite definite expression introduced by the classical logic in the corresponding case is meaningless.

23 An outsaying is said to be a proposition, if it is true or false. An outsaying which is neither true nor false is said to be senseless.

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Notes and Queries

The purpose of this department is to facilitate the communication of research topics in the history and philosophy of logic. Contributions should be written up in a coherent form (rather than as rough notes, say) and with detailed references, in order to facilitate their appraisal. Especially welcome are answers or addenda to material already published in the department. Requests to publish contributions in the form submitted will be honoured as far as possible, but rewriting may be needed (in the light of other contributions which overlap, for example, and in conformity with house style). The proposed published version of a contribution will be sent to its author before publication, where his work will be acknowledged. Contributions should be sent to the editor of the department: J. Corcoran, Department of Philosophy, State University of New York at Buffalo, Buffalo, New York 14260, U.S.A.

From Categoricity to Completeness

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1. Categoricity of postulate sets and equivalence of mathematical systems

Current study of axiomatic method presupposes concepts and results of string theory and set theory. But axiomatic method was vigorously pursued in the last half of the nineteenth century without explicit reference to the theory of strings, which was not axiomatized until the 1930s (independently by Tarski *1934* and by Hermes *1938*). For historical remarks on string theory see Corcoran, Frank and Maloney *1974*. Moreover, although it is natural to read modern set-theoretic ideas in early methodological work, it is clear that early postulate theorists such as Dedekind, Peano, Hilbert, Huntington and Veblen were not working in any formal set-theoretic framework. Indeed, close correspondence between modern precise explications of methodological concepts and imprecise formulations found in the writings of early postulate theorists suggests that the creators of modern mathematical logic, particularly Carnap, Tarski and Church, may have been guided by the goal of putting the early ideas on a more mathematically precise basis.

In the following discussion the term 'system' is used loosely as in Dedekind *1888* for a complex of objects, functions, relations, etc. and *not* for a complex of propositions, symbols, etc. Such constructions as 'axiom system' which violate this conventon are avoided.

By the late 1880s mathematicians had distinguished mathematical systems from propositions referring to the contents of the systems. Dedekind *1888* considered systems $<D, f>$, where D is a class and f is a function from D into itself, and he had isolated the idea of an isomorphism between two such systems. He apparently